Verification of High-Level Transformations with Inductive Refinement Types

Ahmad Salim Al-Sibahi  
IT University of Copenhagen  
University of Copenhagen  
Skanned.com  
Denmark  
ahmad@[di.ku.dk,skanned.com]

Thomas P. Jensen  
Inria Rennes  
France  
thomas.jensen@inria.fr

Aleksandar S. Dimovski  
IT University of Copenhagen  
Denmark  
Mother Teresa University, Skopje  
Macedonia  
aleksandar.dimovski@unt.edu.mk

Andrzej Wąsowski  
IT University of Copenhagen  
Denmark  
wasowski@itu.dk

Abstract

High-level transformation languages like Rascal include expressive features for manipulating large abstract syntax trees: first-class traversals, expressive pattern matching, backtracking and generalized iterators. We present the design and implementation of an abstract interpretation tool, Rabit, for verifying inductive type and shape properties for transformations written in such languages. We describe how to perform abstract interpretation based on operational semantics, specifically focusing on the challenges arising when analyzing the expressive traversals and pattern matching. Finally, we evaluate Rabit on a series of transformations (normalization, desugaring, refactoring, code generators, type inference, etc.) showing that we can effectively verify stated properties.

CCS Concepts  •  Theory of computation → Program verification; Program analysis; Abstraction; Functional constructs; Program schemes; Operational semantics; Control primitives;  •  Software and its engineering → Translator writing systems and compiler generators; Semantics;

Keywords  transformation languages, abstract interpretation, static analysis

1 Introduction

Transformations play a central role in software development. They are used, amongst others, for desugaring, model transformations, refactoring, and code generation. The artifacts involved in transformations—e.g., structured data, domain-specific models, and code—often have large abstract syntax, spanning hundreds of syntactic elements, and a correspondingly rich semantics. Thus, writing transformations is a tedious and error-prone process. Specialized languages and frameworks with high-level features have been developed to address this challenge of writing and maintaining transformations. These languages include Rascal [31], Stratego/XT [11], TXL [15], Uniplate [34] for Haskell, and Kiama [46] for Scala. For example, Rascal combines a functional core language supporting state and exceptions, with constructs for processing of large structures.

```
1 public Script flattenBlocks(Script s) {
2   solve(s) {
3       s = bottom-up visit(s) {
4           case stmtList: [xs,block(ys),*zs] =>
5             xs + ys + zs
6       }
7   }
8   return s;
9 }
```

Figure 1. Transformation in Rascal that flattens all nested blocks in a statement.
To rule out errors in transformations, we propose a static analysis for enforcing type and shape properties, so that target transformations produce output adhering to particular shape constraints. For our PHP example, this would include:

- The transformation preserves the constructors used in the input: does not add or remove new types of PHP statements.
- The transformation produces flat statement lists, i.e., lists that do not recursively contain any block.

To ensure such properties, a verification technique must reason about shapes of inductive data—also inside collections such as sets and maps—while still maintaining soundness and precision. It must also track other important aspects, like cardinality of collections, which interact with target language operations including pattern matching and iteration.

In this paper, we address the problem of verifying type and shape properties for high-level transformations written in Rascal and similar languages. We show how to design and implement a static analysis based on abstract interpretation. Concretely, our contributions are:

1. An abstract interpretation-based static analyzer—Rascal ABstract Interpretation Tool (Rabit)—that supports inferring types and inductive shapes for a large subset of Rascal.
2. An evaluation of Rabit on several program transformations: refactoring, desugaring, normalization algorithm, code generator, and language implementation of an expression language.
3. A modular design for abstract shape domains, that allows extending and replacing abstractions for concrete element types, e.g. extending the abstraction for lists to include length in addition to shape of contents.

Together, these contributions show feasibility of applying abstract interpretation for constructing analyses for expressive transformation languages and properties. We proceed by presenting a running example in Sect. 2. We introduce the key constructs of Rascal in Sect. 3. Section 4 describes the modular construction of abstract domains. Sections 5 to 8 describe abstract semantics. We evaluate the analyzer on realistic transformations, reporting results in Sect. 9. Sections 10 and 11 discuss related papers and conclude.

2 Motivation and Overview

Verifying types and state properties such as the ones stated for the program of Fig. 1 poses the following key challenges:

- The programs use heterogeneous inductive data types, and contain collections such as lists, maps and sets, and basic data such as integers and strings. This complicates construction of the abstract domains, since one shall model interaction between these different types while maintaining precision.
- The traversal of syntax trees depends heavily on the type and shape of input, on a complex program state, and involves unbounded recursion. This challenges the inference of approximate invariants in a procedure that both terminates and provides useful results.
- Backtracking and exceptions in large programs introduce the possibility of state-dependent non-local jumps. This makes it difficult to statically calculate the control flow of target programs and have a compositional denotational semantics, instead of an operational one.

Figure 2 presents a small pedagogical example using visitors. The program performs expression simplification by traversing a syntax tree bottom-up and reducing multiplications by constant zero. We now survey the analysis techniques contributed in this paper, explaining them using this example.
**Inductive refinement types** Rabbit works by inferring an inductive refinement type representing the shape of possible output of a transformation given the shape of its input. It does this by interpreting the simplification program abstractly, considering all possible paths the program can take for values satisfying the input shape (any expression of type `Expr` in this case). The result of running Rabbit on this case is:

```
success cst (Nat) ∨ var (str) ∨ mult (Expr’, Expr’)
fail cst (Nat) ∨ var (str) ∨ mult (Expr’, Expr’)
```

where `Expr’ = cst (suc (Nat)) ∨ var (str) ∨ mult (Expr’, Expr’).

We briefly interpret how to read this type. The bar `∨` denotes a choice between alternative constructors. If the input was rewritten during traversal (success, the first line) then the resulting syntax tree contains no multiplications by zero. All multiplications may only involve `Expr’`, which disallows the zero constant at the top level. Observe this in the last alternative `mult (Expr’, Expr’)` that contains only expressions of type `Expr’`, which in turn only allows multiplications by constants constructed using `suc (Nat)` (that is ≥ 1). If the traversal failed to match (fail, the second line), then the input did not contain any multiplication by zero to begin with and so does not the output, which has not been rewritten.

The success and failure happen to be the same for our example, but this is not necessarily always the case. Keeping separate result values allows retaining precision throughout the traversal, better reflecting concrete execution paths. We now proceed discussing how Rabbit can infer this shape using abstract interpretation.

**Abstractly interpreting traversals** The core idea of abstractly executing a traversal is similar to concrete execution: we recursively traverse the input structure and rewrite the values that match target patterns. However, because of abstraction we must make sure to take into account all applicable paths. Figure 3 shows the execution tree of the traversal on the simplification example (Fig. 2) when it starts with shape `mult (cst (Nat), cst (Nat))`. Since there is only one constructor, it will initially `recurse` down to traverse the contained values (children) creating a new recursion node (yellow, light shaded) in the figure (ii) containing the left child `cst (Nat)`. Since there is only one constructor, it will initially `recurse` down to traverse the contained values (children) creating a new recursion node (yellow, light shaded) in the figure (ii) containing the left child `cst (Nat)`, and then recurse again to create a node (iii) containing `Nat`. Observe here that `Nat` is an abstract type with two possible constructors (`zero`, `suc (-)`) and it is unknown at time of abstract interpretation, which of these constructors we have. When Rabbit hits a type or a choice between alternative constructors, it explores each alternative separately creating new `partition` nodes (blue, darker). In our example we partition the `Nat` type into its constructors `zero` (node iv) and `suc (Nat)` (node v). The zero case now represents the first case without children and we can run the visitor operations on it. Since no pattern matches zero it will return a fail zero result indicating that it has not been rewritten. For the `suc (Nat)` case it will try to recurse down to `Nat` (node vi) which is equal to (node iii). Here, we observe a problem: if we continue our traversal algorithm as is, we will not terminate and get a result. To provide a terminating algorithm we will resort to using `trace memoization`.

**Partition-driven trace memoization** The idea is to detect the paths where execution recursively meets similar input, merging the new recursive node with the similar previous one, thus creating a loop in the execution tree [41, 43]. This loop is then resolved by a fixed-point iteration.

In Rabbit, we propose `partition-driven trace memoization`, which works with potentially unbounded input like the inductive type refinements that are supported by our abstraction. We detect cycles by maintaining a `memoization map` which for each type—used for partitioning—stores the last traversed value (input) and the last result produced for this value (output). This memoization map is initialized to map all types to the bottom element (⊥) for both input and output. The evaluation is modified to use the memoization map, so it checks on each iteration the input `i` against the map:

- If the last processed refinement type representing the input `i’` is greater than the current input (`i ⊑ i’`), then it uses the corresponding output; i.e., we found a hit in the memoization map.
- Otherwise, it will merge the last processed and current input refinement types to a new value `i” = i’ ∨ i`, update the memoization map and continue execution with `i”`. The operation `∨` is called a `widening`; it ensures that the result is an upper bound of its inputs, i.e., `i’ ⊑ i” ⊑ i` and that the merging will eventually terminate for the increasing chain of values. The memoization map is updated to map the general type of `i”` (not refined, for instance `Nat`) to map to a pair (`i”, 0`), where the first component denotes the new input `i”` refinement type and the second component denotes...
the corresponding output of refinement type; initially, 
the widening operator \( \oplus \) is set to \( \bot \) and then changed to the result of executing
input \( \gamma' \) repeatedly until a fixed-point is reached.

We demonstrate the trace memoization and fixed-point iteration procedures on \( \text{Nat} \) in Fig. 4, beginning with the leftmost tree. The expected result is \( \text{fail} \) \( \text{Nat} \), meaning that no pattern has matched, no rewrite has happened, and a value of type \( \text{Nat} \) is returned, since the simplification program only introduces changes to values of type \( \text{Expr} \).

We show the memoization map inside a framed orange box. The result of the widening is presented below the memoization map. In all cases the widening in Fig. 4 is trivial, as it happens against \( \bot \). The final line in node 1 stores the value \( \text{zero} \text{Nat} \) produced by the previous iteration of the traversal, to establish whether a fixed point has been reached (\( \bot \) initially).

**Trace partitioning** We partition [39] the abstract value \( \text{Nat} \) along its constructors: zero and suc (\( \cdot \)) (Fig. 4). This partitioning is key to maintain precision during the abstract interpretation. As in Fig. 3, the left branch fails immediately, since no pattern in Fig. 2 matches zero. The right branch descends into a new recursion over \( \text{Nat} \), with an updated memoization table. This run terminates, due to a hit in the memoization map, returning \( \bot \). After returning, the value of suc \( \text{Nat} \) should be reconstructed with the result of traversing the child \( \text{Nat} \), but since the result is \( \bot \) there is no value to reconstruct with, so \( \bot \) is just propagated upwards. At the return to the last widening node, the values are joined, and widen the previous iteration result \( \text{zero} \text{Nat} \) (the dotted arrow on top). This process repeats in the second and third iterations, but now the reconstruction in node 3 succeeds: the child \( \text{Nat} \) is replaced by \( \text{zero} \) and fail suc \( \text{zero} \) is returned (dashed arrow from 3 to 1). In the third iteration, we join and widen the following components (cf. \( \text{zero} \text{Nat} \) and the dashed arrows incoming into node 1 in the rightmost column):

\[
\text{zero} \lor \text{suc} \text{zero} \lor (\text{zero} \sqcup \text{suc} \text{zero} \sqcup \text{zero}) = \text{Nat}
\]

Here, the used widening operator [17] accelerates the convergence by increasing the value to represent the entire type \( \text{Nat} \). It is easy to convince yourself, by following the same recursion steps as in the figure, that the next iteration, using \( \text{zero} \text{Nat} \) will produce \( \text{Nat} \) again, arriving at a fixed point. Observe, how consulting the memoization map, and widening the current value accordingly, allowed us to avoid infinite recursion over unfoldings of \( \text{Nat} \).

**Nesting fixed point iterations.** When inductive shapes (e.g., \( \text{Expr} \)) refer to other inductive shapes (e.g., \( \text{Nat} \)), it is necessary to run nested fixed-point iterations to solve recursion at each level. Figure 5 returns to the more high-level fragment of the traversal of \( \text{Expr} \) starting with \( \text{mult} \) \( \text{cst} \text{Nat} \), \( \text{cst} \text{Nat} \) as in Fig. 3. We follow the recursion tree along nodes 5, 6, 7, 8, 9, 10, 9, 6 with the same rules as in Fig. 4. In node 10 we run a nested fixed point iteration on \( \text{Nat} \), already discussed in Fig. 4, so we just include the final result.

**Type refinement.** The output of the first iteration in node 6 is \( \text{fail} \text{cst} \text{Nat} \), which becomes the new \( \text{zero} \text{Nat} \), and the second iteration begins (to the right). After the widening the input is partitioned into \( e \) (node 7) and \( \text{cst} \text{Nat} \) (node elided). When the second iteration returns to node 7 we have the following reconstructed value: \( \text{mult} \text{cst} \text{Nat} \text{Nat} \). Contrast this with lines 6-7 in Fig. 2, to see that running the abstract value against this pattern might actually produce success. In order to obtain precise result shapes, we refine the input values when they fail to match a pattern. Our abstract interpreter produces a refinement of the type, by running it through the pattern matching, giving:

\[
\text{success} \text{cst} \text{Nat} \\
\text{fail} \text{mult} (\text{cst} (\text{suc} \text{Nat})), \text{cst} (\text{suc} \text{Nat}))
\]

The result means, that if the pattern match succeeds then it produces an expression of type \( \text{cst} \text{Nat} \). More interestingly, if the matching failed neither the left nor the right argument of \( \text{mult} \) \( \cdot \) could have contained the constant \( \text{zero} \)—the interpreter captured some aspect of the semantics of the program by refining the input type. Naturally, from this point on the recursion and iteration continues, but we shall abandon the example, and move on to formal developments.

3 Formal Language

The presented technique is meant to be general and applicable to many high-level transformation languages. However, to keep the presentation concise, we focus on few key constructs from Rascal [31], relying on the concrete semantics from Rascal Light [2].

We consider algebraic data types \( (a t) \) and finite sets \( (s e t t) \) of elements of type \( t \). Each algebraic data type \( a t \) has a set of unique constructors. Each constructor \( k (t) \) has a fixed set of typed parameters. The language includes sub-typing, with void and value as bottom and top types respectively.

\[
t \in \text{Type} ::= \text{void} \mid \text{set} (t) \mid \text{at} \mid \text{value}
\]

We consider the following subset of Rascal expressions: From left to right we have: variable access, assignments, sequencing, constructor expressions, set literal expressions, matching failure expression, and bottom-up visitors:

\[
e ::= x \in \text{Var} \mid x = e \mid e; e \mid k(e) \mid \{e\} \mid \text{fail} \mid \text{visit} e\text{cs}
\]

\[
cs ::= \text{case} p \Rightarrow e
\]

Visitors are a key construct in Rascal. A visitor \( e \text{cs} \) traverses recursively the value obtained by evaluating \( e \) (any combination of simple values, data type values and collections). During the traversal, case expression \( cs \) are applied to the nodes, and the values matching target patterns are rewritten. We will discuss a concrete subset of patterns \( p \)
Further in Sect. 6. For brevity, we only discuss the bottom-up visitors in the paper. However, Rabit (Sect. 9) supports all visitor strategies of Rascal.

**Notation** We write $\langle x, y \rangle \in f$ to denote the pair $(x, y)$ such that $x \in \text{dom } f$ and $y = f(x)$. Abstract semantic components, sets, and operations are marked with a hat: $\hat{a}$. A sequence of $e_1, \ldots, e_n$ is contracted using an underlining $\underbar{e}$. The empty sequence is written by $\perp$, and concatenation of sequences $e_1$ and $e_2$ is written $\underbar{e} \cdot e_2$. Notation is lifted to sequences in an intuitive manner: for example given a sequence $\underbar{e}$, the value $e_i$ denotes the $i$th element in the sequence, and $\overline{\underbar{e}}$ denotes the sequence $\overline{e_1; \ldots; e_n}$. The empty sequence is written by $\perp$.

**4 Abstract Domains**

Our abstract domains are designed to allow modular composition. Modularity is key for transformation languages, which manipulate a large variety of kinds of values. The design allows easily replacing abstract domains for particular types of values, as well as adding support for new value types. We want to construct an abstract value domain $\hat{sv} \in \text{ValueShape}$ which captures inductive refinement types of form:

$$\hat{a}t' = k_1(\hat{sv}_1) \cdots k_n(\hat{sv}_n)$$

where each value $\hat{sv}_i$ can possibly recursively refer to $\hat{a}t'$. Below, we define abstract domains for sets, data types and recursively defined domains.
The modular domain design generalizes parameterized domains [16] to follow a design inspired by the modular construction of types and domains [7, 14, 44]. The idea is to define domains parametrically—i.e. in the form $\langle \hat{E} \rangle$—so that abstract domains for subcomponents are taken as parameters, and explicit recursion is handled separately. We use standard domain combinators [52] to combine the various domains into our target abstract value domain.

**Set shape domain** Let $\text{Set}(E)$ denote the domain of sets consisting of elements taken from $E$. We define abstract finite sets using abstract elements $\{ \hat{e} \}_{[l,u]}$ from a parameterized domain $\text{SetShape}(\hat{E})$. The component from the parameter domain $(\hat{e} \in \hat{E})$ represents the abstraction of the shape of elements, and a non-negative interval component $[l; u] \in \text{Interval^+}$ is used to abstract over the cardinality (so $l, u \in \mathbb{R}^+$ and $l \leq u$). The abstract set element acts as a reduced product between $\hat{e}$ and $[l; u]$ and thus the lattice operations follow directly.

Given a concretization function for the abstract content domain $\gamma_{\text{DS}}(E) \subseteq \text{SetShape}(\hat{E}) \rightarrow \varphi(E)$, we can define a concretization function for the abstract set shape domain to possible finite sets of concrete elements $\gamma_{\text{DS}} \in \text{SetShape}(\hat{E})$:

$$\gamma_{\text{DS}}(\{ \hat{e} \}_{[l,u]}) = \{ es | es \subseteq \gamma_{\text{E}}(\hat{e}) \land |es| \in \gamma_{\text{T}}([l; u]) \}$$

**Example 4.1.** Let $\text{Interval}$ be a domain of intervals of integers (a standard abstraction over integers). We can concretize abstract elements from $\text{SetShape}(\text{Interval})$ to a set of possible sets of integers from $\varphi(\text{Set}(\hat{Z}))$ as follows:

$$\gamma_{\text{DS}}(\{[42; 43]_{[1;2]}\}) = \{\{42\}, \{43\}, \{42, 43\}\}$$

**Data shape domain** Inductive refinement types are defined as a generalization of refinement types [23, 42, 54] that inductively constrain the possible constructors and the content in a data structure. We use a parameterized abstraction of data types $\text{DataShape}(\hat{E})$, whose parameter $\hat{E}$ abstracts over the shape of constructor arguments:

$$\hat{d} \in \text{DataShape}(\hat{E}) = \{ \bot_{\text{DS}} \} \cup \{ k_1(e_1) \ldots k_n(e_n) | e_i \in \hat{E} \} \cup \{ \top_{\text{DS}} \}$$

We have the least element $\bot_{\text{DS}}$ and top element $\top_{\text{DS}}$ elements—respectively representing no data types value and all data type values—and otherwise a non-empty choice between unique (all different) constructors of the same algebraic data type $k_1(e_1) \ldots k_n(e_n)$ (shortened $k(e)$). We can treat the constructor choice as a finite map $k_1 \mapsto e_1, \ldots, k_n \mapsto e_n$, and then directly define our lattice operations point-wise.

Given a concretization function for the concrete content domain $\gamma_{\text{E}}(E) \subseteq \text{SetShape}(\hat{E}) \rightarrow \varphi(E)$, we can create a concretization function for the data shape domain $\gamma_{\text{DS}} \in \text{DataShape}(\hat{E}) \rightarrow \varphi(\text{Data}(E))$:

We presented the required components for abstracting individual types, and now all that is left is putting everything together. We construct our value shape domain using choice and recursive domain equations:

$$\text{ValueShape} = \text{SetShape}(\text{ValueShape}) \uplus \text{DataShape}(\text{ValueShape})$$

Similarly, we have the corresponding concrete shape domain:

$$\text{Value} = \text{Set}(\text{Value}) \uplus \text{Data}(\text{Value})$$

We then have a concretization function $\gamma_{\text{DS}} \in \text{ValueShape} \rightarrow \varphi(\text{Value})$, which follows directly from the previously defined concretization functions.
Abstract store domain Tracking assignments of variables is important since matching variable patterns depends on the value being assigned in the store:

\[ \tilde{\sigma} \in \text{Store} = \text{Var} \rightarrow \{\text{true, false}\} \times \text{ValueShape} \]

For a variable \( x \) we get \( \tilde{\sigma}(x) = (b, \tilde{v}_s) \) where \( b \) is true if \( x \) might be unassigned, and false otherwise (when \( x \) is definitely assigned). The second component, \( \tilde{v}_s \) is a shape approximating a possible value of \( x \).

We lift the orderings and lattice operations point-wise directly from the target value domain:

\[ \tilde{\sigma} \in \text{Store} \rightarrow \text{ValueShape} \]

Abstract result domain Traditionally, abstract control flow is handled using a collecting denotational semantics with continuations, or by explicitly constructing a control flow graph. These methods are non-trivial to apply for a rich language like Rascal, especially considering backtracking, exceptions and data-dependent control flow introduced by visitors. A nice side-effect of Schmidt-style abstract interpretation is that it allows handling abstraction of control flow directly.

We model different type of results—successes, pattern match failures, errors directly in a ResSet domain which keeps track of possible results with each its own separate store. Keeping separate stores is important to maintain precision around different paths:

\[ \text{Res} \in \text{ResType} := \text{success} \mid \text{exres} \]

\[ \text{exres} := \text{fail} \mid \text{error} \]

\[ \text{Res} \in \text{ResSet} = \text{ResType} \rightarrow \text{ResVal} \times \text{Store} \]

The lattice operations are lifted directly from the target domain and store domains. We define the concretization function \( y_{\text{Store}} : \text{Store} \rightarrow \varphi(\text{Store}) \) as:

\[ y_{\text{Store}}(\tilde{\sigma}) = \left\{ \begin{array}{l}
\forall x, b, \tilde{v}_s, \tilde{\sigma}(x) = (b, \tilde{v}_s) \Rightarrow \\
(\neg b \Rightarrow x \in \text{dom } \sigma) \\
\land (x \in \text{dom } \sigma \Rightarrow \sigma(x) \in y_{\text{Res}}(\tilde{v}_s))
\end{array} \right. \]

Abstract Semantics A distinguishing feature of Schmidt-style abstract interpretation is that the derivation of abstract operational rules from a given concrete operational semantics is systematic and to a large extent mechanisable [9, 43]. The creative work is therefore reduced to providing abstract definitions for conditions and semantic operations such as pattern matching, and defining trace memoization strategies for non-structurally recursive operational rules, to finitely approximate an infinite number of concrete traces and produce a terminating static analysis.

Figure 6 relates the concrete evaluation judgment (left) to the abstract evaluation judgment (right) for Rascal expressions. Both judgements evaluate the same expression \( e \). The abstract evaluation judgment abstracts the initial concrete store \( \sigma \) with an abstract store \( \tilde{\sigma} \). The result of the abstract evaluation is a finite result set \( \tilde{\text{Res}} \), abstracting over possibly infinitely many concrete result values \( \text{rest resv} \) and stores \( \sigma' \). \( \tilde{\text{Res}} \) maps each result type \( \text{rest} \) to a pair of abstract result value \( \tilde{\text{resv}} \) and abstract store \( \tilde{\sigma'} \), i.e.:

\[ \tilde{\text{Res}} = [\text{rest}_1 \mapsto (\tilde{\text{resv}}_1, \tilde{\sigma}_1), \ldots, \text{rest}_n \mapsto (\tilde{\text{resv}}_n, \tilde{\sigma}_n)] \]

There is an important difference in how the concrete and abstract semantic rules are used. In a concrete operational semantics a language construct is usually evaluated as soon as the premises of a rule are satisfied. When evaluating abstractly, we must consider all applicable rules, to soundly over-approximate the possible concrete executions. To this end, we introduce a special notation to collect all derivations with the same input \( i \) into a single derivation with output \( O \) equal to the join of the individual outputs:

\[ \{ i \Rightarrow O \} \triangleq O = \bigcup \{ o | i \Rightarrow o \} \]

Let’s use the operational rules for variable accesses to illustrate the steps in Schmidt-style translation of operational rules. The concrete semantics contains two rules for variable accesses, E-V-S for successful lookup, and E-V-Er for producing errors when accessing unassigned variables:

\[ \text{E-V-S} \quad x \in \text{dom } \sigma \quad x; \sigma \leadsto \text{success } \sigma(x); \sigma \]

\[ \text{E-V-Er} \quad x \notin \text{dom } \sigma \quad x; \sigma \leadsto \text{error } \sigma \]

We follow three steps, to translate the concrete rules to abstract operational rules:

1. For each concrete rule, create an abstract rule that uses a judgment for evaluation of a syntactic form, e.g., AE-V-S and AE-V-Er for variables.
2. Replace the concrete conditions and semantic operations with the equivalent abstract conditions and semantic operations for target abstract values, e.g. \( x \in \text{dom } \sigma \) with \( \tilde{\sigma}(x) = (b, \tilde{v}_s) \) and a check on \( b \). We obtain two execution rules:

\[ e; \sigma \leadsto \text{resv} ; \tilde{\sigma}' \]

\[ x; \sigma \leadsto \tilde{\text{resv}} \land \tilde{\text{resv}} = y_{\text{Res}}(\tilde{\sigma})(\text{rest resv}) \land \tilde{\sigma}' = y_{\text{Store}}(\tilde{\sigma}) \]
The possible shapes of the result value depend on the pair assigned to \( x \) in the abstract store. If the value shape of \( x \) is \( \bot \), we drop the success result from the result set. The following examples illustrate the possible outcome result shapes:

<table>
<thead>
<tr>
<th>Assigned Value</th>
<th>Result Set</th>
<th>Rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\sigma}(x) = (b, \bot) )</td>
<td>{ }</td>
<td>AE-V-S</td>
</tr>
<tr>
<td>( \tilde{\sigma}(x) = (b, [1; 3]) )</td>
<td>{ success ( \mapsto ) ( ([1; 3], \tilde{\sigma}) ) }</td>
<td>AE-V-S</td>
</tr>
<tr>
<td>( \tilde{\sigma}(x) = (t, \bot) )</td>
<td>{ error ( \mapsto ) ( (\cdot, \tilde{\sigma}) ) }</td>
<td>AE-V-S, AE-V-ER</td>
</tr>
<tr>
<td>( \tilde{\sigma}(x) = (t, [1; 3]) )</td>
<td>{ success ( \mapsto ) ( ([1; 3], \tilde{\sigma}) ), error ( \mapsto ) ( (\cdot, \tilde{\sigma}) ) }</td>
<td>AE-V-S, AE-V-ER</td>
</tr>
</tbody>
</table>

It is possible to translate the operational semantics rules for other basic expressions using the presented steps (see Appendix B). The core changes are the ones moving from checks of definiteness to checks of possibility. For example:

- Checking that evaluation of \( e \) has succeeded, requires that the abstract semantics uses \( e; \bar{\tilde{\sigma}} \mapsto \tilde{\sigma} \) and \( (\bar{\tilde{\sigma}}, (\tilde{\sigma}, \bar{\tilde{\sigma}}')) \in \tilde{\mathcal{R}} \), as compared to \( e; \tilde{\sigma} \mapsto \text{success}; \bar{\tilde{\sigma}}; \tilde{\sigma}' \) in the concrete semantics.
- Typing \( \bar{\tilde{\sigma}} \) now requires that the abstract judgments \( \bar{\tilde{\sigma}} \vdash t \) and \( \bar{\tilde{\sigma}} \vdash t' \). In particular, type \( t \) is an abstract subtype of type \( t' \) \( (t \preceq t') \) if there is a subtype \( t'' \) of \( t \) \( (t'' \prec t) \) that is also a subtype of \( t' \) \( (t'' \prec t') \). This implies that \( t \preceq t' \) and \( t \not\preceq t' \) are non-exclusive.
- To check whether a particular constructor is possible, we use the abstract auxiliary function \( \text{unfold}(\bar{\tilde{\sigma}}, t) \) which produces a refined value of type \( t \) if possible—splitting alternative constructors for data type values—and additionally produces error if the value is possibly not an element of \( t \).

6 Pattern Matching

Expressive pattern matching is key feature of high-level transformation languages. Rabit handles the full Rascal pattern language including type-based matching and deep pattern matching. For brevity, we discuss a subset, including variables \( x \), constructor patterns \( k(p) \), and set patterns \( \{\star p\} \):

\[
p := x | k(p) | \{\star p\}
\]

Rascal allows non-linear matching where the same variable \( x \) can be mentioned more than once: all values matched against \( x \) must have equal values for the match to succeed. Each set pattern contains a sequence of sub-patterns \( \star p \); each sub-pattern in the sequence is either an ordinary pattern \( p \) matched against a single set element, or a star pattern \( \star x \) to be matched against a subset of elements. Star patterns can backtrack when pattern matching fails because of non-linear variable references, or when explicitly triggered by the fail expression.

This expressiveness poses challenges for developing an abstract interpreter that is not only sound, but is also sufficiently precise to prove interesting properties. The key aspects of Rabit in handling pattern matching is how we maintain precision by refining input values on pattern matching successes and failures.

6.1 Satisfiability semantics for patterns

We begin by defining what it means that a (concrete/abstract) value matches a pattern. Figure 7a shows the concrete semantics for patterns. In the figure, \( \rho \) is a binding environment:

\[
\rho \in \text{BindingEnv} = \text{Var} \rightarrow \text{Value}
\]

A value \( v \) matches a pattern \( p \) \( (v \models p) \) iff there exists a binding environment \( \rho \) and a function \( k(v) \) of the same constructor whose subcomponents \( v \) match the sub-patterns \( p \) consistently in the same binding environment \( \rho \). A variable \( x \) matches exactly the value it is bound to in the binding environment \( \rho \). A set pattern \( \{\star p\} \) accepts any set of values \( \{v\} \) such that an associative-commutative arrangement of the sub-values \( v \) matches the sequence of sub-patterns \( \star p \) under \( \rho \).

A value sequence \( \bar{v} \) matches a pattern sequence \( \star \bar{p} \) \( (\bar{v} \models \star \bar{p}) \) if there exists a binding environment \( \rho \) such that \( \text{dom} \rho = \text{vars}(\bar{p}) \) and \( \bar{v} \models \star \bar{p} \). An empty sequence of patterns \( \epsilon \) accepts an empty sequence of values \( \epsilon \). A sequence starting \( \bar{p} \) with an ordinary pattern \( p \) matches any non-empty sequence of values \( v, v' \) where \( \bar{v} \) matches \( p \) and \( v' \) matches \( \star \bar{p} \) consistently under the same binding environment \( \rho \). A sequence \( \star x, \star \bar{p} \) works analogously but it splits the value sequence in two \( v, v' \), such that \( x \) is assigned to \( v \) in \( \rho \) and \( v' \) matches \( \star \bar{p} \) consistently in \( \rho \).

**Example 6.1.** We revisit the running example to understand how the data type values are matched. We consider
matching the following set of expression values:

\[ \{ \text{mult} (\text{cst} (\text{zero})), \text{cst} (\text{cst} (\text{zero}))) \} \]

against the pattern \( p = \{ \text{mult} (x, y), \star w, x \} \) in the environment \( \rho = \{ x \mapsto \text{cst} (\text{zero}), y \mapsto \text{cst} (\text{zero}), w \mapsto \{ \} \} \).

The matching argument is as follows:

\[
\{ \{ w \} \} \models^* \rho p \quad \text{if} \quad \{ \text{mult} (x, y), \star w, x \} \models^* \text{cst} (\text{zero}) \quad \text{and} \quad \text{cst} (\text{zero}) \models^* \text{mult} (x, y)
\]

Similarly, the second matches as follows:

\[
\text{cst} (\text{zero}) \models^* \star w, x \quad \text{iff} \quad \text{cst} (\text{zero}) \models^* \{ \}
\]

6.2 Computing pattern matches

The declarative satisfiability semantics of patterns, albeit quite clean, is unfortunately not directly computable. In Rabit, we rely on an abstract operational semantics (see appendix A), translated from the concrete operational pattern matching semantics [2], using similar technique to the one presented in Sect. 5. The interesting ideas are in the refining semantic operators used, which we will discuss further.

Semantic operators with refinement Since Rascal supports non-linear matching, it becomes necessary to merge environments computed when matching sub-patterns to check whether a match succeeds or not. In abstract interpretation, we can refine the abstract environments when merging for each possibility. Consider when merging two abstract environments, where some variable \( x \) is assigned to \( \sim \) in one, and \( \sim' \) in the other. If \( \sim' \) is possibly equal to \( \sim \), we refine both values using this equality assumption \( \sim \equiv \sim' \).

Here, we have that abstract equality is defined as the greatest lower bound if the value is non-bottom, i.e. \( \sim \equiv \sim' \) \( \equiv \{ \sim'' \mid \sim'' \equiv \sim \wedge \sim'' \neq \bot \} \). Similarly, we can also refine both values if they are possibly non-equal \( \sim \not\equiv \sim' \). Here, abstract inequality is defined using relative complements:

\[
\sim \not\equiv \sim' \equiv \{ \sim'', \sim''' \mid \sim'' \equiv \sim' \wedge \sim''' \equiv \sim \wedge \sim''' \neq \bot \} \cup \{ \sim, \sim'' \mid \sim'' \equiv \sim' \wedge \sim'' \neq \bot \}
\]

In our abstract domains, the relative complement (\( \cdot \)) is limited. We heuristically define it for interesting cases, and otherwise it degrades to identity in the first argument (no refinement). There are however useful cases, e.g., for excluding unary constructors \( \text{successor} (\text{Nat}) \mid \text{zero} \) or at the end points of a lattice \( \{ 1; 10 \} \mid \{ 1; 2 \} = \{ 3; 10 \} \).

Similarly, for matching against a constructor pattern \( k(p) \), the core idea is that we should be able to partition our value
Abstract interpretation and static program analysis in general perform fixed-point calculation for analysing unbounded loops and recursion. In Schmidt-style abstract interpretation, the main technique to handle recursion is trace memoization \([41, 43]\). The core idea of trace memoization is to detect non-structural re-evaluation of the same program element, i.e., when the evaluation of a program element is recursively dependent on itself, like a while-loop or traversal.

The main challenge when recursing over inputs from infinite domains, is to determine when to merge recursive paths together to correctly over-approximate concrete executions. We present an extension that is still terminating, sound and, additionally, allows calculating results with good precision. The core idea is to partition the infinite input domain using a finite domain of elements, and on recursion degrade input values using previously met input values from the same partition. We assume that all our domains are lattices with a widening operator. Consider a recursive operational semantics judgment \( i \implies o \), with \( i \) being an input from domain Input, and \( o \) being the output from domain Output. For this judgment, we associate a memoization map \( M \in \Pi_{\text{Input}} \rightarrow \text{Input} \times \text{Output} \) where \( \Pi_{\text{Input}} \) is a finite partitioning domain that has a Galois connection with our actual input, i.e., Input \( \iff \) \( \Pi_{\text{Input}} \). The memoization map keeps track of the previously seen input and corresponding output for values in the partition domain. For example, for input from our value domain Value we can use the corresponding type from the domain Type as input to the memoization map.\(^2\) So for values 1 and [2; 3] we would use \( \text{int} \), while for \( \text{mult}((\text{Expr}, \text{Expr})) \) we would use the defining data type \( \text{Expr} \).

\(^2\)Provided that we bound the depth of type parameters of collections.
We perform a fixed-point calculation over the evaluation of input $i$. Initially, the memoization map $\hat{M}$ is $\lambda \alpha.(\bot, \bot)$, and during evaluation we check whether there was already a value from the same partition as $i$, i.e., $\alpha_{\hat{P}I}(i) \in \text{dom } \hat{M}$. At each iteration, there are then two possibilities:

**Hit** The corresponding input partition key is in the memoization map and a less precise input is stored, so $\hat{M}(\alpha_{\hat{P}I}(i)) = (i', o')$ where $i \not\in_{\text{input}} i'$. Here, the output value $o$ that is stored in the memoization map is returned as result.

**Widen** The corresponding input partition key is in the memoization map, but an unrelated or more precise input is stored, i.e., $\hat{M}(\alpha_{\hat{P}I}(i)) = (i'', o'')$ where $i \not\in_{\text{input}} i''$. In this case we continue evaluation but with a widened input $i' = i''\triangledown_{\text{input}} (i'' \sqcup i)$ and an updated map $\hat{M}' = [\alpha_{\hat{P}I}(i) \mapsto (i', o_{\text{prec}})]$. Here, $o_{\text{prec}}$ is the output of the last iteration for the fixed-point calculation for input $i'$, and is assigned $\bot$ on the initial iteration.

Intuitively, the technique is terminating because the partitioning is finite, and widening ensures that we reach an upper bound of possible inputs in a finite number of steps, eventually getting a hit. The fixed-point iteration also uses widening to calculate an upper bound, which similarly finishes in a number of steps. The technique is sound because we only use output for previous input that is less precise; therefore our function is continuous and a fixed-point exists.

## 9 Experimental Evaluation

We demonstrate the ability of Rabit to verify type and inductive shape properties, using five transformation programs across various applications. Three programs are classic examples, and two are extracted from open source projects.

**Negation Normal Form** (NNF) transformation [27, Section 2.5] is a classical rewrite of a propositional formula to combination of conjunctions and disjunctions of literals, so negations appear only next to atoms. An implementation of this transformation should guarantee the following:

- **P1** Implication is not used as a connective in the result
- **P2** All negations in the result are in front of atoms

**Rename Struct Field** (RSF) refactoring changes the name of a field in a struct, and that all corresponding field access expressions are renamed correctly as well:

- **P3** Structure should not define a field with the old field name
- **P4** No field access expression to the old field

**Desugar Oberon-0** (DSO) transformation [6, 53], translates for-loops and switch-statements to while-loops and nested if-statements, respectively.

- **P5** for should be correctly desugared to while
- **P6** switch should be correctly desugared to if
- **P7** No auxiliary data in output

### Code Generation for Glagol (G2P)

(G2P) a DSL for REST-like web development, translated to PHP for execution. We are interested in the part of the generator that translates Glagol expressions to PHP, and the following properties:

- **P8** Output only simple PHP expressions for simple Glagol expression inputs
- **P9** No unary PHP expressions if no sign marks or negations in Glagol input

**Mini Calculational Language** (MCL) a programming language text-book [45] implementation of a small expression language, with arithmetic and logical expressions, variables, if-expressions, and let-bindings. The implementation contains an expression simplifier (larger version of running example in Fig. 2), a type inference procedure, an interpreter and a compiler.

- **P10** Simplification procedure produces a simplified expression with no additions with 0, multiplications with 1 or 0, subtractions with 0, logical expressions with constant Boolean operands, and if-expressions with constant Boolean conditions.
- **P11** Arithmetic expressions with no variables have type int and no type errors
- **P12** Interpreting expressions with no integer constants and let’s gives only Boolean values
- **P13** Compiling expressions with no if’s produces no goto’s and if instructions
- **P14** Compiling expressions with no if’s produces no labels and does not change label counter

All these transformations satisfy the following criteria:

1. They are formulated by an independent source,
2. They can be translated in relatively straightforward manner to our subset of Rascal, and
3. They exercise important constructs, including visitors and the expressive pattern matching

We have ported all these programs to Rascal Light.

### Threats to validity.

The programs are not selected randomly, thus it is hard to generalize the results for other transformations. We mitigated this by selecting transformations that are realistic and vary in authors, programming style and purpose. While translating the programs to Rascal Light, we strived to minimize the amount of changes, but generally bias cannot be ruled out entirely.

### Implementation.

We have implemented the abstract interpreter in a prototype tool, Rabit, for all of Rascal Light following the process described in sections 5 to 8. This required handling additional aspects, not discussed in the paper:

1. Possibly undefined values
2. Extended result state with more Control flow constructs, backtracking, exceptions, loop control, and

---

https://github.com/BulgariaPHP/glagol-dsl
3. Fine-tuning memoization strategies to the different looping constructs and recursive calls

By default, we use the top element $\top$ for the types specified as input. The user can specify the initial data-type refinements, store and parameters, to get a more precise result for target function to be abstractly interpreted. The output of the tool is the abstract result value set of abstractly interpreting target function, the resulting store state and the set of relevant inferred data-type refinements.

The implementation extends standard regular tree grammar operations \cite{1, 17}, to handle the recursive equations for the expressive abstract domains, including base values, collections and heterogeneous data types. We use a more precise partitioning strategy for trace memoization when needed, which also takes the set of available constructors into account for data types. The source code of our implementation, including subject transformations, is freely available.

\section*{Results}

We ran the experiments using Scala 2.12.2 on a 2012 Core i5 MacBook Pro. Table 1 summarizes the size of the programs, the runtime of the abstract interpreter, and whether the properties have been verified. Since we verify the results on the abstract shapes, the programs then are shown to be correct for all possible concrete inputs satisfying the given properties. We remark that all programs use the high-level expressive features of Rascal and are thus significantly more succinct than comparable code in general purpose languages.

The runtime, varying from single seconds to less than a minute, is reasonable. All, but two, properties were successfully verified. The reason that our tool runs slower on

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Transformation & LOC & Runtime [s] & Property & Verified \\
\hline
NNF & 15 & 7.3 & P1 & ✓ \\
 & & & P2 & ✓ \\
RSF & 35 & 6.0 & P3 & X \\
 & & & P4 & ✓ \\
DSO & 125 & 25.0 & P5 & ✓ \\
 & & & P6 & ✓ \\
 & & & P7 & X \\
G2P & 350 & 1.6 & P8 & ✓ \\
 & & & 3.5 & P9 & ✓ \\
 & & & 1.6 & P10 & ✓ \\
 & & & 0.7 & P11 & ✓ \\
 & & & 0.6 & P12 & ✓ \\
 & & & 0.9 & P13 & ✓ \\
 & & & & P14 & ✓ \\
MCL & 298 & & & \\
\hline
\end{tabular}
\caption{Time and success rate for analyzing programs and properties presented earlier this section. Time is the median of five runs. If the same time is reported for multiple properties, then they could be verified on the same input}
\end{table}

\footnote{https://github.com/itu-square/Rascal-Light}

\section*{Figure 8. Initial and inferred refinement types for NNF}

\begin{Verbatim}
data FIn = and(FIn, FIn) | atom(str) | neg(FIn) \\
| imp(FIn, FIn) | or(FIn, FIn) \\
data FOut = and(FOut, FOut) | atom(str) \\
| neg(atom(str)) | or(FOut, FOut)
\end{Verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Initial and inferred refinement types for NNF}
\end{figure}

the DSO transformation than those with more lines of code (G2P and MCL), is that it contains many nested traversals expressed as function calls: our analysis is interprocedural but handles function calls by inlining which can lead to some overhead during analysis.

Lines 1–2 in Fig. 8 show the input refinement type $\text{FIn}$ for the normalization procedure. The inferred inductive output type $\text{FOut}$ (lines 4–5) specifies that the implication is not present in the output ($\text{P1}$), and negation only allows atoms as subformulae ($\text{P2}$). In fact, Rabit inferred a precise characterization of negation normal form as an inductive data type.

\section*{10 Related Work}

We start with discussing techniques that could be used to make Rabit infer more precise shapes and verify properties like $\text{P3}$ and $\text{P7}$. To verify $\text{P3}$, we need to be able to relate field names to their corresponding definitions in the field definition map of a class, which is not possible using the presented non-relational abstract domains. Relational abstract interpretation \cite{35} allows specifying such constraints that relate values across different variables, and even inside and across substructures \cite{13, 26, 33}. For a concrete input of $\text{P7}$, we know that the number of auxiliary data elements decreases on each iteration, but this information is lost in our abstraction of data structures. A possible solution could be to allow \textit{abstract attributes} that extract additional information about the abstracted structures \cite{10, 37, 49}. For $\text{P7}$, a generalization of the multiset abstraction \cite{36} for data types, could be useful to track e.g., the auxiliary statement count, and show that they decrease using multiset-ordering \cite{22} like in term rewriting. Other techniques \cite{4, 13, 51} support inferring inductive relational properties for general data-types—e.g, binary tree property—but require a pre-specified structure to indicate the places where refinement can happen.

Cousot and Cousot \cite{18} present a general framework for modularly constructing program analyses, but it requires a language with a compositional control flow which Rascal does not have. Toubhans, Rival and Chang \cite{40, 50} develop a modular domain design for pointer-manipulating programs supporting a rich set of fixed data abstractions, whereas our domain construction focuses on providing automated inference of inductive refinement types based on pure heterogeneous data-structures.
There are similarities between our work and verification techniques based on program transformation (e.g., [21, 32]) like partial evaluation [29] and supercompilation [48]. Our systematic exploration of execution rules for abstraction is similar to unfolding, and our use of widening is similar to folding. The main difference between the two techniques is that abstract interpretation mainly focuses on capturing rich domains and performing widening at syntactic program points, whereas program transformation based techniques often rely on symbolic inputs and perform folding dynamically on the semantic execution graph during specialization. We believe that there could be benefits for the communities, to explore combinations of these two approaches in the future.

Definitional interpreters have been suggested as a technique for building compositional abstract interpreters [20]. The idea is to rely on a monad transformer stack to share the implementation of the concrete and abstract interpreters. We believe that our interpreter would benefit by being written in such style, which complements our modular domain construction well. To ensure termination they rely on a caching algorithm, similar to ordinary finite input trace memoization [41]. Similarly, Van Horn and Might [28] present a systematic framework to abstract higher-order functional languages with effects and complex control flow. They rely on store-allocated continuations within abstract machines to handle recursion, which is then kept finite during abstraction to ensure a terminating analysis. Our technique focused on providing a more precise widening based on the abstract input value, which was necessary for verifying the required properties in our evaluation. We believe that it could be useful to look into abstract machine-based abstractions in the future, in the case that higher-order transformation languages need to be handled.

Garrigue [24, 25] presents algorithms for typing pattern matching on polymorphic variant types in OCaml, where the set of constructors for a data type is not fixed in advance. The theory is useful since it supports inferring simple recursive shapes of programs, but it has its limitations: inference is syntactic and exact, and it is unclear how to generalize it to work with the rich pattern matching constructs and heterogeneous visitors. Haskell supports analysing coverage of its pattern matching language, that includes generalized algebraic data types (GADTs) and Boolean constraints [30]. While general Haskell function calls can occur in the Boolean constraints, the analysis treats them shallowly as function symbols; some covering pattern matches that depend on particular semantics of called functions, will be marked falsely as non-exhaustive. Modern SMT solvers supports reasoning with inductive functions defined over algebraic data-types [38]. The properties they can verify are very expressive, and include inductive semantic properties. The exact techniques employed are not very scalable, and encoding a complex transformation directly would not finish verifying even simple properties within reasonable time. Possible constructor analysis [5] has been used to calculate the actual dependencies of a predicate and make flow-sensitive analyses more precise. This is a type of shape analysis that works with complex data-types and arrays, but only captures the prefix of the target structures.

Techniques for model transformation verification based on static analysis [19] have been suggested, but are currently focused on verification of rule errors based on types and undefinedness. Symbolic execution has previously been suggested [3] as a way to validate high-level transformation programs. However, that work targets test generation rather than verification of properties. Semantic typing [8, 12] has been used to infer recursive type and shape properties for language with high-level constructs for querying and iteration. The languages considered are however small calculi compared to the supported subset of Rascal we consider, and our evaluation is significantly more extensive.

11 Conclusion

Our goal was to use abstract interpretation to give a solid semantic foundation for analyzing programs in modern high-level transformation languages. To this end we have designed and formalized a Schmidt-style abstract interpreter, including partition-driven trace memoization which works with infinite input domains. This worked well for a language like Rascal with complex control flow, and can be adapted work for similar languages that have an operational semantics. The proposed modular construction of abstract domains was vital for handling a language of this scale and complexity.

We implemented the interpreter as a tool, Rabit, which supports a non-trivial subset of Rascal, containing key features: several traversal strategies, expressive pattern matching, backtracking, exceptions and control operators, and generalized looping constructs. We evaluated Rabit on classical transformations and on examples selected from open source projects, showing it allows verification of a series of sophisticated type and shape properties for these transformations.

A Operational Pattern Matching

Computing Pattern Matching The judgements are presented in Fig. 9 for both the concrete and abstract rules. Consider the abstract (top-left) judgement: a value $v$ matches a pattern $p$, given a store $\sigma$, producing a sequence of binding environments $\rho$. The binding environments form a sequence, since multiple concrete environments, say $\rho_1$ and $\rho_2$, can make $v$ match against $p$, i.e., $\sigma \models_{\rho_1} p$ and $\sigma \models_{\rho_2} p$.

Backtracking using the fail-expression, allows the programmer to explore a different assignment from the sequence of environments, until no possible assignment is left.

For an ordinary pattern $p$ (top) the abstraction relation is direct: an abstract store $\hat{\sigma}$ abstracts a concrete store $\sigma$
and a value shape \( \tilde{vs} \) abstracts a concrete value \( v \). The notable change is that the abstract semantics uses a set of abstract binding environments \( \tilde{Q} \subseteq Store \times ValueShape \times BindingEnv \) that not only abstracts over the sequence of concrete binding environments \( p \), but also, for each abstract binding environment stores the corresponding refinement of the input abstract store \( \tilde{\sigma} \) and the corresponding refinement of the matched value shape \( \tilde{vs} \) according to the matched pattern.

For sequences of set sub-patterns \( p \), the sequence of concrete values \( v \) is abstracted by two components: the shape of values \( \tilde{vs} \) and an interval approximating the length of the value sequence \([l; u]\). Both of these values are refined as a result of the matching, which is captured by the abstract binding environment \( \tilde{Q} \) (of the same type as for the simple patterns), since we treat the value refined as the abstract set containing the values of the given shape and of given cardinality. The concrete semantics of set sub-patterns also contains a backtracking state \( \tilde{v} \) which is not used in the abstract semantics, because the abstraction of set elements is coarse and we thus abstractly consider all possible subset assignments at the same time (joining instead of backtracking).

**Operational Rules** We will show how refinement is calculated by the abstract operational semantics by presenting some of key rules for abstract pattern matching. Rascal also allows non-linear pattern matching against assigned store variables, and it is possible to use this information for refining the input store and abstract value. In the AP-V-U rule we match the variable to the value shape and restrict the abstraction for the variable value to match the pattern. The binding environment does not change as the name is already bound in the store. In the AP-V-F rule, the matching fails (⊥), and then we learn that the value shape in the store should be refined to something that does not match.

\[
\text{AP-V-U} \quad \tilde{\sigma}(x) = (b, \tilde{vs}) \quad \tilde{vs}' \in (\tilde{vs} = \tilde{vs}') \quad \tilde{\sigma'} = \tilde{\sigma}[x \leftrightarrow (\text{ff}, \tilde{vs}'')] \\
\tilde{\sigma} x \vdash \tilde{vs} \quad \tilde{\sigma'} = \tilde{\sigma}[x \leftrightarrow (\text{ff}, \tilde{vs}'')] \\
\tilde{\sigma} x \vdash \tilde{vs} \quad \tilde{\sigma'} = \tilde{\sigma}[x \leftrightarrow (\text{ff}, \tilde{vs}'')] \\
\text{AP-V-F} \quad \tilde{\sigma}(x) = (b, \tilde{vs}) \quad \tilde{vs}' \notin \tilde{vs} \quad \tilde{\sigma'} = \tilde{\sigma}[x \leftrightarrow (\text{ff}, \tilde{vs}'')] \\
\tilde{\sigma} x \vdash \tilde{vs} \quad \tilde{\sigma'} = \tilde{\sigma}[x \leftrightarrow (\text{ff}, \tilde{vs}'')] \\
\tilde{\sigma} x \vdash \tilde{vs} \quad \tilde{\sigma'} = \tilde{\sigma}[x \leftrightarrow (\text{ff}, \tilde{vs}'')] \\
\]

We also show the AP-V-B (abstract pattern-variable-bind) rule which simply binds the variable in the binding environment, assuming that it is possibly not assigned in the store (a free name).

\[
\text{AP-V-B} \quad \tilde{\sigma}(x) = (tt, \tilde{vs}) \\
\tilde{\sigma} x \vdash \tilde{vs} \quad \tilde{\sigma}[x \leftrightarrow (tt, \tilde{vs})] \quad \tilde{\sigma}, x \mapsto \tilde{vs} \\
\]

If our matched abstract value possibly contains the pattern constructor \( k \) (AP-C-S rule: abstract pattern-constructor-success) we produce an abstract value with \( k \) containing the sub-values refined against constructor sub-patterns:

\[
\text{data at } = \cdots \mid k(t) \mid \cdots \\
\text{(success } k'(\tilde{vs}') \text{)} \in \text{unfold}(\tilde{vs}, \text{at}) \\
\tilde{\sigma} + p \vdash p_n \quad \tilde{\sigma}(\tilde{vs}) = \tilde{\sigma}(\tilde{vs}) \quad \tilde{\sigma}(\tilde{vs}) = \tilde{\sigma}(\tilde{vs}) \\
\text{AP-C-S} \quad \tilde{\sigma} + k(p) \vdash \tilde{vs} \quad \tilde{\sigma}, \text{exclude}(\tilde{vs}, k, \perp) \\
\]

The total function \( \text{merge} \) unifies assignments from two binding environments point-wise by names, taking the greatest lower bound of shapes to combine bindings for a name. It yields bottom for the entire result if at least one of the pointwise meets yields bottom (shapes for at least one name are not reconcilable). Otherwise, we try to refine the matched value to exclude the pattern constructor in the AP-C-F rules:

\[
\text{data at } = \cdots \mid k(t) \mid \cdots \\
\text{(success } k'(\tilde{vs}') \text{)} \in \text{unfold}(\tilde{vs}, \text{at}) \\
\tilde{\sigma} + k(p) \vdash \tilde{vs} \quad \tilde{\sigma}, \text{exclude}(\tilde{vs}, k, \perp) \\
\text{AP-C-F} \quad \tilde{\sigma} + k(p) \vdash \tilde{vs} \quad \tilde{\sigma}, \text{exclude}(\tilde{vs}, k, \perp) \\
\]

For set patterns, the refinement happens by pattern matching set sub-patterns.

\[
\text{success } \{\tilde{vs}\}_{[l; u]} \in \text{unfold}(\tilde{vs}, \text{set(value)}) \\
\tilde{\sigma} + p \vdash \tilde{vs}, [l; u] \\
\text{AP-S-S} \quad \tilde{\sigma} + p \vdash \tilde{vs}, [l; u] \\
\]

For example, when it is possible that the abstracted value sequence \( \tilde{vs}, [l; u] \) is empty \( (l = 0) \) and patterned matched against an empty set sub-pattern sequence, we can refine the result to be the empty abstract set \( \{\perp\} \) (rule APL-E-B).

\[
\text{APL-E-B} \quad \tilde{\sigma} + \epsilon \vdash \tilde{vs}, [l; u] \\
\tilde{\sigma} + \epsilon \vdash \tilde{vs}, [l; u] \\
\]

A more complex example is the one where we try to pattern match a potentially non-empty value sequence against a set sub-pattern sequence \( p, p' \) starting with an ordinary pattern (APL-M-P). Here we pattern match against \( p \) and the rest of the sequence \( p' \) and combine the refined results of these matches producing a refinement of the containing set value by combining the refined shapes and increasing the
Verification of High-Level Transformations...

refinement of the length by the set sub-pattern sequence by one.

\[
\begin{align*}
  l \leq u & \quad u \neq 0 \quad \hat{\sigma} \vdash p \overset{?}{=} \hat{\nu}_s \quad \overset{\text{a-match}}{\longrightarrow} \hat{Q}_R' \\
  \hat{\sigma} \vdash \hat{\nu}_s, [l; u - 1] \quad \overset{\text{a-match}}{\longrightarrow} \hat{Q}_R'' \\
  (\hat{\sigma}', \hat{\nu}_s; \hat{Q}') \in \hat{Q}_R' \quad (\hat{\sigma}'', \{\hat{\nu}_s\} [l''; u'']) \in \hat{Q}_R'' \\
  \hat{Q}_R''' = \{(\hat{\sigma} \cap \hat{\sigma}'', \{\hat{\nu}_s \cup \hat{\nu}_s\} [l''+1; u''+1])
\end{align*}
\]

\text{APL-M-F}

\[
\begin{align*}
  \hat{\sigma} \vdash p, \hat{\nu}_s, [l; u] \quad \overset{?}{=} \quad \overset{\text{a-match} + 1}{\longrightarrow} \hat{Q}_R'''
\end{align*}
\]

B  Abstract Semantic Rules

Figures 11 and 12 shows the formal rules for executing the bottom-up visit-expression; we have omitted the collecting rules and some error handling rules to avoid presenting unnecessary details. We will further discuss the ideas behind the rules in a high-level fashion.

**Executing visitors** The evaluation rule for the visit-expression itself is mainly concerned with evaluating the target expression \( e \) to be traversed to a value, and then using a separate traversal relation to rewrite the value recursively with the sequence of cases \( \hat{\nu}_s \). The main item to notice is how it uses the value refined by the case patterns in case of failure (\( \text{AE-Vr-f} \)), turning the result into successful execution (like in our running example in Sect. 2).

**Evaluating Cases** During traversal, the target value will be rewritten with a sequence of cases. The evaluation of a case sequence is straight-forward, iterating through the possible cases, pattern matching against each pattern and executing the corresponding expression when applicable. The main idea is that, when the abstract value fails to match a pattern, the refined value is used to match against the rest of the cases (\( \text{ACS-M-F} \)). This ensures that the order of patterns influences the refinement, leading to a more precise abstract shape that better matches the set of concrete shapes during execution.

Acknowledgments

We would like to thank Paul Klint, Tijs van der Storm, Jurgen Vinju and Davy Landman for discussions on Rascal and its semantics. We would further like to thank Rasmus Møgelberg and Jan Midtgård for discussions on correctness of our recursive shape abstractions. We would like the anonymous reviewers for their comments, especially the one who presented us the link between our style of abstract interpretation and verification techniques in program transformation.

This material is based upon work supported by the Danish Council for Independent Research under Grant No. 0602-02327B and Innovation Fund Denmark under Grant No. 7039-0007B. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the funding agencies.

References


Figure 9. Relating abstract operational semantics (left) to the concrete operational semantics (right).

### Expressions (General)

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x = e; \tilde{\sigma} \rightarrow \tilde{\text{Res}}}$</td>
<td>Assignment Expression</td>
</tr>
<tr>
<td>${e_1; e_2; \tilde{\sigma} \rightarrow \tilde{\text{Res}}}$</td>
<td>Expression Sequences</td>
</tr>
<tr>
<td>${k(e); \tilde{\sigma} \rightarrow \tilde{\text{Res}}}$</td>
<td>Assignment Expression</td>
</tr>
</tbody>
</table>

#### Assignment Expression

$\{x = e; \tilde{\sigma} \rightarrow \tilde{\text{Res}}\}$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e; \tilde{\sigma} \rightarrow \tilde{\text{Res}}$</td>
<td>Assignment Expression</td>
</tr>
</tbody>
</table>

#### Sequencing Expression

$\{e_1; e_2; \tilde{\sigma} \rightarrow \tilde{\text{Res}}\}$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${e_1; e_2; \tilde{\sigma} \rightarrow \tilde{\text{Res}}}$</td>
<td>Sequencing Expression</td>
</tr>
</tbody>
</table>

#### Constructor Expression

$\{k(e); \tilde{\sigma} \rightarrow \tilde{\text{Res}}\}$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${k(e); \tilde{\sigma} \rightarrow \tilde{\text{Res}}}$</td>
<td>Constructor Expression</td>
</tr>
</tbody>
</table>

#### Expression Sequences

$\{e; e' \rightarrow \tilde{\text{Res}}\}$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${e; e' \rightarrow \tilde{\text{Res}}}$</td>
<td>Expression Sequences</td>
</tr>
</tbody>
</table>

### Figure 10. Abstract Operational Semantics Rules for Basic Expressions
### Visit Expression

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e, \bar{\sigma} \xrightarrow{\text{a-expr}} \text{Res}$</td>
<td>(success, $(\hat{v}_s, \hat{\sigma}'')$) $\in \text{Res}$</td>
</tr>
<tr>
<td>$\text{cs; } \bar{v}_s, \hat{\sigma}' \xrightarrow{a-\text{bu-visit}} \text{Res}'$</td>
<td>(success, $(\hat{v}_s', \hat{\sigma}'')$) $\in \text{Res}'$</td>
</tr>
<tr>
<td>$\text{visit } e \text{ cs; } \bar{\sigma} \xrightarrow{\text{a-expr-visit}} [\text{success } \mapsto (\hat{v}_s', \hat{\sigma}')]$</td>
<td>AE-Vr-S</td>
</tr>
<tr>
<td>$e, \bar{\sigma} \xrightarrow{\text{a-expr}} \text{Res}$</td>
<td>(success, $(\hat{v}_s, \hat{\sigma}'')$) $\in \text{Res}$</td>
</tr>
<tr>
<td>$\text{cs; } \bar{v}_s, \hat{\sigma}' \xrightarrow{a-\text{bu-visit}} \text{Res}'$</td>
<td>(success, $(\hat{v}_s', \hat{\sigma}'')$) $\in \text{Res}'$</td>
</tr>
<tr>
<td>$\text{visit } e \text{ cs; } \bar{\sigma} \xrightarrow{\text{a-expr-visit}} [\text{success } \mapsto (\hat{v}_s', \hat{\sigma}')]$</td>
<td>AE-Vr-F</td>
</tr>
</tbody>
</table>

### Bottom-up Traversal of Single Value

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{(vs}', \text{cvvs}) \in \text{children}(\hat{v}_s)$</td>
<td>$\text{cs; cvvs; } \bar{\sigma} \xrightarrow{a-\text{bu-visit}<em>} \text{Res}</em>$ (success, $(\hat{v}_s', \hat{\sigma}')$) $\in \text{Res}*$</td>
</tr>
<tr>
<td>$\text{reconvs}' \text{ using cvvs to RCRes}$</td>
<td>$\text{cs; cvvs; } \bar{\sigma} \xrightarrow{a-\text{cases}} \text{Res}'$</td>
</tr>
<tr>
<td>$\text{(vs}', \text{cvvs}) \in \text{children}(\hat{v}_s)$</td>
<td>$\text{cs; cvvs; } \bar{\sigma} \xrightarrow{a-\text{bu-visit}<em>} \text{Res}</em>$ (success, $(\hat{v}_s', \hat{\sigma}')$) $\in \text{Res}*$</td>
</tr>
<tr>
<td>$\text{reconvs}' \text{ using cvvs to } \text{success } \mapsto \text{vcombine}$</td>
<td>$\text{cs; cvvs; } \bar{\sigma} \xrightarrow{a-\text{cases}} \text{Res}'$</td>
</tr>
</tbody>
</table>

### Bottom-up Traversal of Children

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{cs; e; } \bar{\sigma} \xrightarrow{a-\text{bu-visit}* \mapsto \text{go}}$</td>
<td>$\text{fail } \mapsto (e, \bar{\sigma})$</td>
</tr>
<tr>
<td>$\text{cs; vs; } \bar{v}_s, \bar{\sigma} \xrightarrow{a-\text{bu-visit}* \mapsto \text{go}}$</td>
<td>$\text{Res}$</td>
</tr>
<tr>
<td>$\text{cs; (vs; [0; u]); } \bar{\sigma} \xrightarrow{a-\text{bu-visit}* \mapsto \text{go}}$</td>
<td>$\text{fail } \mapsto ([\bot], 0, \bar{\sigma})$</td>
</tr>
<tr>
<td>$\text{cs; vs; } \bar{v}_s, \bar{\sigma} \xrightarrow{a-\text{bu-visit}* \mapsto \text{go}}$</td>
<td>$\text{Res}$</td>
</tr>
<tr>
<td>$\text{cs; vs; } \bar{v}_s, \bar{\sigma} \xrightarrow{a-\text{bu-visit}* \mapsto \text{go}}$</td>
<td>$\text{Res}$</td>
</tr>
</tbody>
</table>

### Figure 11. Selected Abstract Operational Semantics Rules for Traversal


<table>
<thead>
<tr>
<th>Case Sequence</th>
<th>Case Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACS-E</td>
<td>ACS-M-O</td>
</tr>
<tr>
<td>$\sigma \vdash p \overset{\alpha}{\Rightarrow} \widehat{\sigma}$</td>
<td>$\rho^\alpha; e; \widehat{\sigma} \overset{a\text{-case}}{\Rightarrow} \widehat{\rho}$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>$\widehat{\sigma} \vdash p \overset{\alpha}{\Rightarrow} \widehat{\sigma}$</td>
<td>$(\widehat{\sigma}', \widehat{\sigma}, \widehat{\rho}) \in \widehat{\eta}$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>$\rho^\alpha; e; \widehat{\sigma} \overset{a\text{-case}}{\Rightarrow} \widehat{\rho}$</td>
<td>$(\widehat{\rho}, \widehat{\sigma}, \widehat{\rho}) \in \widehat{\eta}$</td>
</tr>
<tr>
<td>ACS-M-F</td>
<td>ACS-M-F</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>$\widehat{\sigma} \vdash p \overset{\alpha}{\Rightarrow} \widehat{\sigma}$</td>
<td>$(\widehat{\rho}, \widehat{\sigma}, \widehat{\rho}) \in \widehat{\eta}$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>$\rho^\alpha; e; \widehat{\sigma} \overset{a\text{-case}}{\Rightarrow} \widehat{\rho}$</td>
<td>$(\widehat{\rho}, \widehat{\sigma}, \widehat{\rho}) \in \widehat{\eta}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACS-E</td>
<td>ACS-M-O</td>
</tr>
<tr>
<td>$\perp ; e; \widehat{\sigma} \overset{a\text{-case}}{\Rightarrow} \widehat{\rho}$</td>
<td>$\rho^\alpha; e \overset{a\text{-expr}}{\Rightarrow} \widehat{\rho}$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>$\widehat{\sigma} \vdash p \overset{\alpha}{\Rightarrow} \widehat{\sigma}$</td>
<td>$\widehat{\rho} \vdash (\widehat{\rho}, \widehat{\sigma}) \overset{\alpha\text{-case}}{\Rightarrow} \widehat{\rho}$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>$\rho^\alpha; e \overset{a\text{-expr}}{\Rightarrow} \widehat{\rho}$</td>
<td>$(\widehat{\rho}, \widehat{\sigma}) \overset{\alpha\text{-case}}{\Rightarrow} \widehat{\rho}$</td>
</tr>
</tbody>
</table>

**Figure 12.** Selected Abstract Operational Semantic Rules for Traversal (Cont.)


